The Art of Concession in General Lotto Games*

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Abstract. Success in adversarial environments often requires investment into additional resources in order to improve one's competitive position. But, can intentionally decreasing one's own competitiveness ever provide strategic benefits in such settings? In this paper, we focus on characterizing the role of "concessions" as a component of strategic decision making. Specifically, we investigate whether a player can gain an advantage by either conceding budgetary resources or conceding valuable prizes to an opponent. While one might naïvely assume that the player cannot, our work demonstrates that – perhaps surprisingly – concessions do offer strategic benefits when made correctly. In the context of General Lotto games, we first show that neither budgetary concessions nor value concessions can be advantageous to either player in a 1-vs.-1 scenario. However, in settings where two players compete against a common adversary, we find opportunities for one of the two players to improve her payoff by conceding a prize to the adversary. We provide a set of sufficient conditions under which such concessions exist.

Keywords: Game theory · Resource allocation · General Lotto games

1 Introduction

Strategic advantages are often held by competitors that possess more budgetary resources that can be invested in more advanced technology, research, or surveillance in order to improve one's competitive position against opponents. Such factors are central to many domains that feature competitive interactions, such as airport security [20,27], wildlife protection [30], market economics [18], and political campaigning [26]. In this paper, we analyze "concessions" as a viable, alternative component of strategic decision-making in adversarial environments. In particular, we seek to identify whether or not conceding one's competitive position can ever be advantageous. Intuitively, concessions would appear to be contradictory to the conventional wisdom on how to gain a strategic advantage, e.g., investing in more resources or information, as concessions weaken one's competitive position. Nonetheless, this paper demonstrates that such intuition is false as appropriately chosen concessions can often be strategically beneficial.

Within the framework of General Lotto games, we study two types of concessions. The first type, which we term *budgetary concessions*, involves willingly reducing one's resource budget. The act of "money burning" serves as an analogy for this type of concession. The second type of concession, which we term *battlefield concessions*, involves voluntary non-participation on a non-zero

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valued battlefield. An appropriate analogy for this type of concession from economoics is "market abandonment". In these scenarios, we assume that concessions are announced to all other players immediately after they are made, such that the other players can respond strategically to the modified competitive environment.

General Lotto games, Colonel Blotto games, and other contest models offer a flexible framework to generate basic insights about the interplay between a competitor's performance guarantees and the amount of resources reserved for competition [3,6,9,13,14,23]. In common formulations, two opposing players have limited resource budgets to allocate to multiple battlefields. A player wins a battlefield and its associated value if she sends more resources than her opponent. To study the role of concessions as a strategic component, we continue this section with a brief overview of General Lotto games and describe our extensions that allow us to study concessions under this model. We also provide a summary of our contributions, namely, the identification of settings where concessions are beneficial. Finally, we draw connections between our work and the related literature.

1.1 General Lotto games with concessions

The General Lotto game is played between two opposing players, A and B, who each have a limited budget of resources $X_A, X_B \geq 0$. The players compete over a set of n battlefields $\mathcal{B} = \{1, \ldots, n\}$, where a player wins a battlefield $b \in \mathcal{B}$ and its value $v_b \geq 0$ by allocating more resources to b than the opponent. The players make moves simultaneously (i.e., a one-shot game). Each player can use randomized allocations such that the resources spent do not exceed its limited budget in expectation. We denote an instance of the General Lotto game with $\text{GL}(X_A, X_B, \mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ is the vector of battlefield valuations. The equilibrium strategies and payoffs in any instance are characterized in the existing literature [13,15], and we reproduce these in Section 2.

We consider the following extension in order to study the strategic role of concessions in General Lotto games: One of the players, say player B, has the option to either voluntarily reduce her own resource budget, or to voluntarily withdraw completely from a chosen battlefield, before engaging with A in the resulting General Lotto game. Specifically, B selects one of the following options:

- *Budgetary concession:* Player B selects some nonzero value $x \in [0, X_B]$, whereupon her resource budget is reduced from X_B to $X_B - x$.
- *Battlefield concession:* Player B selects a battlefield $b \in \mathcal{B}$. The value of the battlefield, v_b, is immediately awarded to player A.

The complete competitive interaction between A and B occurs in two stages. In Stage 0, B decides to concede either budgetary resources or a battlefield to A, as described above. Player B's decision in this stage then becomes binding and common knowledge. Subsequently, in Stage 1, the players engage in the resulting General Lotto game. If a budgetary concession of $x \in [0, X_B]$ was made in Stage 0, the game $\text{GL}(X_A, X_B - x, \mathbf{v})$ is played and the players receive their respective equilibrium payoffs. If a battlefield concession of $b \in \mathcal{B}$ was made in Stage 0, the value v_b is immediately awarded to player A, and the game $\text{GL}(X_A, X_B, \mathbf{v}_{-b})$ is played. Here, $\mathbf{v}_{-b}$ is the vector of valuations for the battlefields $\mathcal{B} \backslash \{b\}$. We say that a player has a *beneficial concession* if there exists any concession such that the player secures a strictly higher payoff than her payoff in the nominal General Lotto game (i.e. without concessions). For example, if player B has a beneficial budgetary concession in the General Lotto game, then there exist parameters $X_A, X_B > 0$, $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ and $x \in [0, X_B]$ such that B's equilibrium payoff is greater in $\text{GL}(X_B - x, X_A, \mathbf{v})$ than in $\text{GL}(X_B, X_A, \mathbf{v})$. Our first contribution is as follows:

Contribution 1. *There never exist concessions of either type that improve a player's payoff in the General Lotto game (Proposition 1).*

1.2 Three-player General Lotto games with concessions

Contribution #1 conforms with the conventional intuition that concessions only ever weaken one's position in competitive scenarios. We thus seek to address whether this phenomenon holds more generally. To that end, we shift our focus to a three-player setting, in which players B and C compete in General Lotto games against a common adversary A over two disjoint sets of battlefields $\mathcal{B}_B, \mathcal{B}_C$ whose valuations are given by the vectors $\mathbf{v}_B, \mathbf{v}_C$, respectively. This formulation was first proposed and studied in [16]. The top diagram in Figure 1a depicts a *nominal* three-player Lotto game (under no concession options). We consider the case where only player B has the option to make concessions. The competitive interaction occurs over three stages as follows, where players' actions become binding and common knowledge in subsequent stages:

- *Stage 0:* Player B decides to make either a budgetary or battlefield concession;
- *Stage 1:* Player A deploys resources $X_{A,B}, X_{A,C} \geq 0$ to the two competitions against B and C, where $X_{A,B} + X_{A,C} \leq X_A$ must be satisfied; and,
- *Stage 2:* Player A engages in the two resulting General Lotto games. If a budgetary concession of $x \in [0, X_B]$ was made in Stage 0, then she plays the game $\text{GL}(X_{A,B}, X_B - x, \mathbf{v}_B)$ against player B. Else, if a battlefield concession of $b \in \mathcal{B}_B$ was made in Stage 0, then she plays $\text{GL}(X_{A,B}, X_B, \mathbf{v}_{B,-b})$ against B, where $\mathbf{v}_{B,-b}$ denotes the vector of valuations for battlefields $\mathcal{B}\backslash\{b\}$. The game $\text{GL}(X_{A,C}, X_C, \mathbf{v}_C)$ is played against player C.

The bottom diagram in Figure 1a depicts the scenario following a battlefield concession. In Stage 1, we assume player A employs an optimal division of resources such that her cumulative payoff from the two General Lotto games in Stage 2 is maximized. Such optimal divisions are characterized in the literature by [16], and we reproduce these results in the forthcoming Section 2. In this three-player setting, we say that B has a *beneficial concession* if there exist any concessions such that B secures a strictly higher payoff than her payoff in the nominal three-player General Lotto game, i.e. if B were to make no concession in Stage 0. Our second contribution is as follows:

Contribution 2. *In three-player General Lotto games, there never exist budgetary concessions that improve a player's payoff (Theorem 1); however, there* do *exist battlefield concessions that can improve a player's payoff. Theorem 2 provides a set of sufficient conditions for when such opportunities are available.*

In the standard, two-player General Lotto game, we show that beneficial concessions do not exist, and, indeed, our result concerning budgetary concessions in the three-player General Lotto game further supports this naïve intuition. However, our results show that beneficial battlefield concessions do exist in three-player General Lotto games, contradicting the conventional wisdom on what constitute viable mechanisms for gaining strategic advantages. More generally, our results suggest that concessions do, in fact, represent reasonable strategic options in competitive interactions.

As our main objective is to establish whether it is possible that beneficial concessions exist in three-player General Lotto games, we consider budgetary concessions of value $x \in [0, X_B]$ and show that $x = 0$ maximizes player B's final payoff. Similarly, in the case of battlefield concessions, instead of considering possible values in the vector $\mathbf{v}_B$, we focus on identifying the battlefield concession value $v \in [0, \Phi_B]$ that maximizes player B's final payoff, where Φ_B is the cumulative value of battlefields in $\mathcal{B}_B$. When $v > 0$, B has a beneficial battlefield concession in any corresponding three-player General Lotto game with $v_b = v$ for some battlefield $b \in \mathcal{B}_B$. In the following example, we identify the occurrence of beneficial battlefield concessions for player B, and the magnitude of B's payoff improvement under various parameterizations of the three-player General Lotto game:

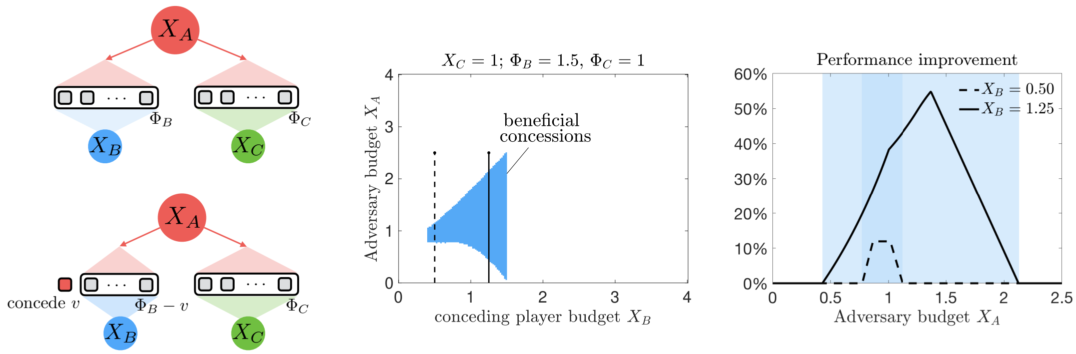
Consider a three-player General Lotto game in which players' initial budget endowments satisfy $X_A, X_B \in [0, 4]$ and $X_C = 1$, and where the cumulative battlefield values in fronts $\mathcal{B}_B$ and $\mathcal{B}_C$ are $\Phi_B = 1.5$ and $\Phi_C = 1$, respectively. For every such game, we compare player B's payoff in the nominal game against her payoff after a battlefield concession of each value $v \in [0, \Phi_B]$, and identify the optimal battlefield concession value v^{opt}. Figure 1b illustrates the regime of initial player budgets in which there exist battlefield concessions of any value $v \in [0, \Phi_B]$ such that player B's payoff in the resulting game is strictly higher than her payoff in the corresponding nominal game (i.e., the regime where there exists a beneficial battlefield concession for B). Figure 1c shows the percentage improvement over player B's payoff in the nominal game associated with conceding the corresponding optimal battlefield concession value v^{opt}. We plot this percentage improvement for $X_A \in [0, 2.5]$, and $X_B = 0.5$ (dashed line) or $X_B = 1.25$ (solid line).

Intuitively, our results illustrate that battlefield concessions in the three-player General Lotto game – if done properly – can redirect more of player A's budget toward player C's set of battlefields, rather than drawing more of A's budget to B's remaining set of battlefields, as the remaining value on B's set of battlefields, $\Phi_B - v^{\text{opt}}$, becomes less of a priority for A. In a sense, the conceding player "appeases" the common adversary by freely offering up a portion of the cumulative battlefield value, and faces less competition as a result. The presence of the additional player C is critical for there to be benefits derived from such concessions. In contrast, budgetary concessions invite A to further pursue her contest against the weakened player B. A budgetary concession reduces B's strength with no change to the cumulative value of the battlefields. This increases the ratio between value and strength on $\mathcal{B}_B$, and leads to A seeking even more value from that front.

1.3 Related Works

A primary line of research in Colonel Blotto games focuses on characterizing its equilibria. Since Borel's initial study [3], many works have advanced this thread over the last one hundred years [2,9,15,17,23,24,28]. However, solutions to the most general settings remain as open problems. As such, there are several variants of the Colonel Blotto game that have been studied extensively, none more so than the General Lotto game [1,13,15,19]. Notably, the players' equilibrium payoffs in the General Lotto games have been fully characterized [13,15]. Due to its tractability, the General Lotto game is often adopted in studies of more complex adversarial environments, including engineering domains such as network security [7,10,25] and the security of cyber-physical systems [5,12].

Our work in this paper is closest to a recent thread in the literature on similar sequential Colonel Blotto and General Lotto games, where players have the option to publicly announce their strategic intentions ahead of play. The three-player General Lotto game was first introduced in [16], who study their own variant model where in Stage 0, players B and C have the opportunity to form an alliance that takes the form of a unilateral budgetary transfer between the players. It is shown that there are cases in which the two players can make unilateral budgetary transfers that are mutually beneficial. Subsequent work in [11,12] considers similar settings where the two players can decide to add battlefields in addition to transferring resources amongst each other. Counter-intuitively, under this model, both the players achieve better payoff if the transfers are publicly announced to their adversary. The authors of [4] identify a sufficient condition for when



(a) (b) (c)

Fig. 1: (a) The top diagram depicts the nominal three player General Lotto game, where the adversary (A) must decide how to divide its endowment to two separate fronts of battlefields, with cumulative values of Φ_B and Φ_C, respectively. The optimal division for A and the resulting payoffs are well-known from the literature [16]. The bottom diagram shows a scenario where player B concedes a battlefield of value $v \in [0, \Phi_B]$. The adversary responds by re-calculating her optimal division based on the modified environment. We seek to answer whether B can benefit from concessions. (b) The parameter region (in blue) where player B has an incentive to concede battlefields. Here, we set $X_C = 1$ and the total valuations of the two fronts are $\Phi_B = 1.5$, $\Phi_C = 1$. (c) The percentage improvement over player B's payoff in the nominal three player game (without concessions) associated with the optimal battlefield concession. We plot the improvements when $X_B = 0.5$ (dashed line) and $X_B = 1.25$ (solid line) for all values $X_A \in [0, 2.5]$, as depicted in (b).

publicly pre-committing resources to battlefields offers strategic advantages in the same three player setting. Pre-commitments are a broader class of concessions, where instead of giving away value, the pre-committing player puts a price in terms of budgetary resources on a battlefield. The pre-commitment of resources is also studied in [29], but in a different context that involves favouritism. In that work, a one-shot Colonel Blotto game is studied where resources are pre-allocated non-strategically over the various battlefields. More broadly, interest in the role of pre-emption and information in contests has popularized the analysis of sequential versions of these game models in which one player leads the strategic interaction and the other follows (see, e.g., [8,21,22]). Though the selection of concessions under our proposed framework is also sequential, we note that the players' strategic interactions still occur simultaneously in the final stage.

The formation of alliances such as those studied in [11,12,16] is often not possible, either because mechanisms for coordination between the agencies are not available or because the agencies' budgets are not directly transferable. In contrast, concessions offer a means for a player to improve her competitive position, even when mutual coordination is not possible. Another notable difference between concessions and alliances is that, while alliances can only lead to mutually beneficial outcomes for the players involved, our results suggest that any benefits derived from concessions by one player must come at an expense to the other.

2 Model

In this section, we review useful background on the standard, two-player General Lotto game, then formalize the three-player General Lotto game model.

2.1 Background on General Lotto games

The standard General Lotto game consists of two players A and B with respective, fixed budgets $X_A, X_B > 0$ competing over the set of n battlefields $\mathcal{B} = \{1, \dots, n\}$ (i.e., front). A player wins on a battlefield b by allocating more budget to b than her opponent, and otherwise loses on b.[3] For each battlefield $b \in \mathcal{B}$, the winning player receives her value $v_b \geq 0$, while the losing player receives zero. Let $\mathbf{v} \in \mathbb{R}^n_{\geq 0}$ denote the vector of battlefield values. An allocation is any vector $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$, where x_b denotes the amount of budget allocated to battlefield b. An admissible strategy for each player $i \in \{A, B\}$ is an n-variate distribution F_i on $\mathbb{R}^n_{\geq 0}$ that satisfies the following budget constraint:

$$\mathbb{E}_{\mathbf{x} \sim F_i}\left[\sum_{b \in \mathcal{B}} x_b\right] \leq X_i. \tag{1}$$

Intuitively, a player may select any distribution over vectors $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ such that the budget expenditure does not exceed her budget *in expectation*. Each player aims to maximize the expected value won over the battlefields. We observe that the game is a two-player, constant-sum game played in a single stage (Stage 1), and that an instance of the game can be succinctly denoted as $\mathrm{GL}(X_A, X_B, \mathbf{v})$. The General Lotto game is a relaxation of the Colonel Blotto game [3], in which the players' allocations must satisfy the budget constraint with probability 1.

The equilibrium characterization of the General Lotto game is well-understood [13,15], and each instance $\mathrm{GL}(X_A, X_B, \mathbf{v})$ is known to admit unique equilibrium payoffs as follows:

Fact 1. *Let* $\mathrm{GL}(X_A, X_B, \mathbf{v})$ *denote an instance of the General Lotto game, and* $\Phi = \sum_{b \in \mathcal{B}} v_b$. *The equilibrium payoff to player* $i \in \{A, B\}$ *is* $\Phi \cdot L(X_i, X_{-i})$, *where*

$$L(X_i, X_{-i}) = \begin{cases} \frac{X_i}{2X_{-i}} & \text{if } X_i \leq X_{-i} \\ 1 - \frac{X_{-i}}{2X_i} & \text{if } X_i > X_{-i}, \end{cases} \tag{2}$$

and $-i \in \{A, B\} \setminus \{i\}$ *is the opposing player.*

As discussed in Section 1.1, concessions in the two player General Lotto game can be considered by introducing an additional stage (Stage 0) that occurs before the players engage in the General Lotto game (Stage 1). Recall that, in Stage 0, player B makes either a budgetary concession or battlefield concession, which then becomes binding and common knowledge before Stage 1 is played. In the following proposition, we show that neither type of concession can ever increase a player's payoff over her payoff in the nominal General Lotto game:

Proposition 1. *Consider the General Lotto game with* $X_A, X_B \geq 0$ *and* $\Phi \geq 0$. *Neither player can benefit from either a budgetary or battlefield concession.*

Proof. We consider the scenario where player B makes either a budgetary or battlefield concession in Stage 0. Since we make no assumption on the players' relative strengths, considering player B's perspective is without loss of generality.

Firstly, from the equilibrium payoffs identified in Fact 1, if player B makes a budgetary concession, i.e., $X'_B \leq X_B$, then it follows that $\Phi \cdot L(X'_B, X_A) \leq \Phi \cdot L(X_B, X_A)$ since, for fixed y, $L(x, y)$ is monotonically increasing in x. Second, and finally, if player B makes a battlefield concession, i.e., $\Phi' \leq \Phi$, then $\Phi' \cdot L(X_B, X_A) \leq \Phi \cdot L(X_B, X_A)$ since L is nonnegative. □

[3] In the case that the players allocate the same amount of budget to a battlefield, the player with higher overall budget is conventionally awarded the win. However, the choice of tie-breaking rule has no effect on equilibrium characterizations of General Lotto games [15], and hence, our results.

2.2 Three-player General Lotto games with concessions

We have shown that concessions cannot provide payoff improvements in the two-player General Lotto game. Thus, we consider the three-player game model proposed in [16] for the remainder of this manuscript. This game consists of players A, B and C with respective budgets $X_A, X_B, X_C > 0$. Player A is engaged in simultaneous General Lotto games against the players B and C over the respective, disjoint fronts $\mathcal{B}_B$ and $\mathcal{B}_C$. The game is played in two stages: in Stage 1, player A allocates her budget between the two fronts; and, in Stage 2, the two resulting General Lotto games are played. In Stage 2, players B and C receive the payoffs from their respective General Lotto games, and player A receives the sum of her expected payoffs from both General Lotto games. An instance of the game can be succinctly denoted as $3\text{GL}(X_A, X_B, X_C, \mathbf{v}_B, \mathbf{v}_C)$, where $\mathbf{v}_i$ denotes the vector of battlefield values in front $\mathcal{B}_i$, $i \in \{B, C\}$. As we have already done with the standard General Lotto game, we propose a variation on the three-player General Lotto model that includes a preliminary stage (Stage 0) in which player B makes either a budgetary or battlefield concession. Below, we formalize the three stages of this variant, which we term the *three-player General Lotto game with concessions*, where it is assumed that the players' actions in each stage become binding and common knowledge in subsequent stages:

- *Stage 0:* Player B selects one of the following concession formats:
  - *Budgetary concession:* Player B discards a portion of her budget $x \in (0, X_B]$; or,
  - *Battlefield concession:* Player B commits to allocating zero budget to a battlefield $b \in \mathcal{B}_B$.
- *Stage 1:* Player A allocates $X_{A,B}, X_{A,C} \geq 0$ of her budget to the fronts $\mathcal{B}_B$ and $\mathcal{B}_C$, respectively, such that $X_{A,B} + X_{A,C} \leq X_A$ holds.
- *Stage 2:* Player A engages players B and C in the two resulting General Lotto games. If B made a budgetary concession of $x \in (0, X_B]$ in Stage 0, then A and B play the game $\text{GL}(X_{A,B}, X_B - x, \mathbf{v}_B)$. Else, if B made a battlefield concession of $b \in \mathcal{B}_B$, then A and B play the game $\text{GL}(X_{A,B}, X_B, \mathbf{v}_{B,-b})$, where $\mathbf{v}_{B,-b}$ denotes the vector of valuations for battlefields $\mathcal{B}_B \setminus \{b\}$. Players A and C play the game $\text{GL}(X_{A,C}, X_C, \mathbf{v}_C)$. Player A's payoff is the sum of her expected payoffs in the two General Lotto games, and of v_b only if player B selected to concede the battlefield b in Stage 0. Each player $i \in \{B, C\}$ receives the expected payoff from her corresponding General Lotto game against A.

In order to identify player B's optimal strategy in Stage 0 of the game we must first understand player A's strategic behaviour in Stage 1. The allocation rule that maximizes A's cumulative payoff in Stage 2 was characterized by Kovenock and Roberson [16]. We summarize this result below:

Fact 2. *Consider Stage 1 of the three-player General Lotto game where the players' budgets are normalized (w.l.o.g.) such that $X_A = 1$ and $X_B, X_C > 0$. Let $\Phi_B, \Phi_C > 0$ denote the cumulative value of non-conceded battlefields in the fronts $\mathcal{B}_B$ and $\mathcal{B}_C$, respectively. Define $\mathcal{R}_{1i}$, $\mathcal{R}_{2i}$, $\mathcal{R}_{3i}$ and $\mathcal{R}_4$, $i \in \{B, C\}$ as the following regions:*

$$\begin{aligned}
\mathcal{R}_{1i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ \textit{s.t.}\ \Phi_i/\Phi_{-i} > \max\{(X_i)^2, 1\}/(X_i X_{-i})\} \\
&\quad \cup \{(X_i, X_{-i}) \ \textit{s.t.}\ X_i < 1 \ \textit{and}\ \Phi_i/\Phi_{-i} = 1/(X_i X_{-i})\} \\
\mathcal{R}_{2i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ \textit{s.t.}\ \Phi_i/\Phi_{-i} > X_i/X_{-i} \ \textit{and}\ 0 < 1 - \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}} \leq X_{-i}\} \\
\mathcal{R}_{3i}(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ \textit{s.t.}\ \Phi_i/\Phi_{-i} \geq X_i/X_{-i} \ \textit{and}\ 1 - \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}} > X_{-i}\} \\
\mathcal{R}_4(\Phi_i, \Phi_{-i}) &:= \{(X_i, X_{-i}) \ \textit{s.t.}\ \Phi_i/\Phi_{-i} = X_i/X_{-i} \ \textit{and}\ X_i + X_{-i} \geq 1\}.
\end{aligned}$$

Player A's optimal allocation $X_{A,i}$ is determined in closed-form as follows:

– *If* $(X_i, X_{-i}) \in \mathcal{R}_{1i}(\Phi_i, \Phi_{-i})$*, then* $X_{A,i} = 1$.
– *If* $(X_i, X_{-i}) \in \mathcal{R}_{2i}(\Phi_i, \Phi_{-i})$*, then* $X_{A,i} = \sqrt{\Phi_i X_i X_{-i}/\Phi_{-i}}$.
– *If* $(X_i, X_{-i}) \in \mathcal{R}_{3i}(\Phi_i, \Phi_{-i})$*, then* $X_{A,i} = \sqrt{\Phi_i X_i}/(\sqrt{\Phi_i X_i} + \sqrt{\Phi_{-i} X_{-i}})$.
– *If* $(X_i, X_{-i}) \in \mathcal{R}_4(\Phi_i, \Phi_{-i})$*, then any* $X_{A,i} \in [1 - X_{-i}, X_i]$ *is optimal,*

where $X_{A,-i} = 1 - X_{A,i}$ *in all the above cases.*

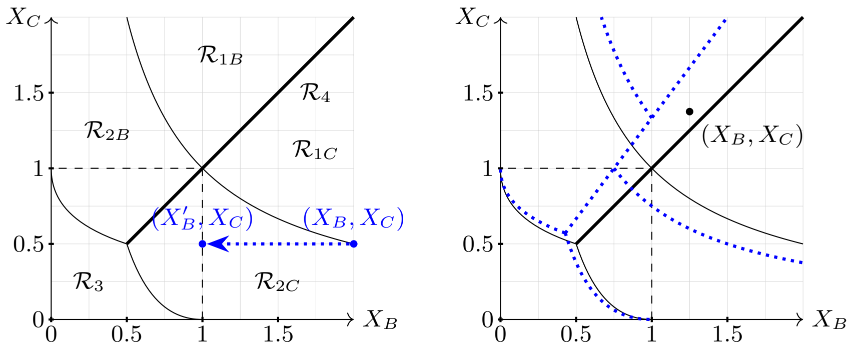
Observe that the result above can be applied in Stage 1, whether or not player B makes a concession. If player B makes no concessions (i.e., the nominal game), then her payoff in Stage 2 is $\Phi_B \cdot L(X_B, X_{A,B})$, where we use $X_{A,B}$ here to denote player A's optimal allocation to the front $\mathcal{B}_B$ in Stage 1 when there is no concession. Otherwise, if player B makes a budgetary concession of $x \in (0, X_B]$, then her payoff in Stage 2 is $\Phi_B \cdot L(X_B - x, X'_{A,B})$, and if player B makes a battlefield concession of $b \in \mathcal{B}_B$, then her payoff in Stage 2 is $(\Phi_B - v_b) \cdot L(X_B, X''_{A,B})$, where we use $X'_{A,B}$ and $X''_{A,B}$ here to denote player A's optimal allocation in Stage 1 to the General Lotto game against player B in response to the budgetary and battlefield concessions, respectively. Crucially, observe that player B translates the point (X_B, X_C) to the left by making budgetary concessions, and alters the parametric regions identified in Fact 2 by making battlefield concessions.

The following observations will be important in the proofs of the forthcoming results:

i. $X_{A,B} > X_B$ holds in regions $\mathcal{R}_{1B}$ (if $X_B < 1$), $\mathcal{R}_{2B}$, $\mathcal{R}_{3B}$ and $\mathcal{R}_{3C}$, while $X_{A,B} \leq X_B$ holds in regions $\mathcal{R}_{1B}$ (if $X_B \geq 1$), $\mathcal{R}_{1C}$, $\mathcal{R}_{2C}$ and $\mathcal{R}_4$.
ii. The closed-form expressions of player A's optimal allocation – and, thus, all of the players' payoffs – are identical in regions $\mathcal{R}_{3B}$ and $\mathcal{R}_{3C}$. Thus, it is equivalent to denote the union of the parametric regions $\mathcal{R}_{3i}(\Phi_i, \Phi_{-i})$, $i \in \{B, C\}$, simply by $\mathcal{R}_3(\Phi_B, \Phi_C)$. Further, note that any point (X_B, X_C) in $\mathcal{R}_3$ must satisfy $X_B + X_C < 1$ since $X_{A,i} > X_i$, $i \in \{B, C\}$.
iii. As shown in Figure 2(a), the point (X_B, X_C) translates to the left in the (X_B, X_C)-plane following a budgetary concession by player B. On the other hand, the regions identified in Fact 2 remain unperturbed.
iv. In contrast, the concession of a battlefield with value $\hat{v}$ does not translate the point (X_B, X_C), but rather modifies the regions identified in Fact 2, as shown in Figure 2(b). In particular, the line $X_C = \Phi_C X_B/\Phi_B$ which serves as the boundary between the regions $\mathcal{R}_{1B} \cup \mathcal{R}_{2B}$ and $\mathcal{R}_{1C} \cup \mathcal{R}_{2C}$ (i.e., the *median line*) rotates counterclockwise about the origin following a battlefield concession by player B. Crucially, since the points on the (X_B, X_C)-plane remain stationary, a point (X_B, X_C) can move from one region to another (i.e., *transit*) following a battlefield concession by either player.

3 Main Results

All of the results in this section focus on concessions in the three-player General Lotto game from the perspective of player B. However, by flipping the players' labels, all the results apply identically to concessions from the perspective of player C. Throughout this section, we refer to concessions that strictly improve the player's payoff above her payoff in the nominal setting as beneficial budgetary and battlefield concessions.



a) Budgetary concession

b) Battlefield concession

Fig. 2: *The regions dividing the possible player budgets* (X_B, X_C) *in Stage 2, as derived in [16] and reviewed in Fact 2.* (a) Illustration of the Stage 1 regions in the three-player General Lotto game with $\Phi_B = \Phi_C$. The solid, black lines depict the borders between the labelled regions. In blue, we depict the impact of a budgetary concession: the point $(X_B, X_C) = (2, 0.5)$ translates to the left to $(X_B - x, X_C) = (1, 0.5)$ after player B makes a budgetary concession of $x = 1$. (b) We depict the impact of a battlefield concession within the same setting as (a), but where B makes a battlefield concession of b with $v_b = \Phi_C/4$. The solid, black lines depict the borders between the regions for no concession (i.e., $\mathcal{R}_j(\Phi_B, \Phi_C)$), while the dotted, blue lines depict the borders between the regions after the concession of b (i.e., $\mathcal{R}_j(\Phi_B - v_b, \Phi_C)$). Observe that all points on the plot, including $(X_B, X_C) = (1.25, 1.375)$, remain stationary, while the regions change. Notably, (X_B, X_C) is in region $\mathcal{R}_{1B}$ if no concession is made, and in $\mathcal{R}_{1C}$ after the concession of battlefield b.

Budgetary concessions. We first focus on budgetary concessions, and show that players cannot improve their payoffs by making such concessions.

Theorem 1. *Consider the three-player General Lotto game with* $X_A = 1$, $X_B, X_C \geq 0$ *and* $\Phi_B, \Phi_C \geq 0$. *Player* B *cannot benefit from a budgetary concession.*

We present the proof of Theorem 1 in Appendix A, for ease of presentation. As the proof is fairly technical, we provide an intuitive interpretation for the reader's convenience. Suppose player B makes a budgetary concession of $x \in (0, X_B]$. Observe that the budgetary concession leaves player B more vulnerable to attacks from player A, since her budget is lowered, but the cumulative value of the battlefields in front $\mathcal{B}_B$ remains unchanged. As a result, the adversary will seek either the same or greater payoff from the front $\mathcal{B}_B$. In the best case, the amount of payoff that the adversary extracts from the front $\mathcal{B}_B$ will stay the same, as is the case if the pairs (X_B, X_C) and $(X_B - x, X_C)$ are both in $\mathcal{R}_{1C}(\Phi_B, \Phi_C)$, i.e., player A still sends no budget to $\mathcal{B}_B$. In all other settings, player B's payoff will strictly decrease after a budgetary concession.

Battlefield concessions. Next, we focus on battlefield concessions. Here, we are concerned with identifying instances in which a battlefield concession is beneficial for player B, i.e., B's resulting payoff is higher than in the nominal game. In particular, we seek conditions on the budgets X_A, X_B, and X_C, and the players' front values Φ_B and Φ_C for which there exists a beneficial battlefield concession. Note here that we are not concerned with the particular vectors of battlefield valuations $\mathbf{v}_B, \mathbf{v}_C$ that constitute each front. As such, we allow player B to have full choice over the conceded value $v \in [0, \Phi_B]$. Our next result identifies sufficient conditions for the existence of beneficial battlefield concessions in any three-player General Lotto game.

Theorem 2. *Consider the three-player General Lotto game with $X_A = 1$, $X_B, X_C \geq 0$ and $\Phi_B, \Phi_C \geq 0$. Let $v^* = \Phi_B - \Phi_C X_B / X_C$. The following conditions characterize sufficient conditions under which player B has a beneficial battlefield concession of value $v_b = v^*$:*

(i) *If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ and $X_B, X_C \geq 1$, then $v^* < \Phi_B/(2X_B)$;*
(ii) *If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$, $X_B \geq 1$ and $X_C < 1$, then*

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C / (\Phi_B - v^*)}}{2X_B}\right] > \Phi_B \cdot \left(1 - \frac{1}{2X_B}\right);$$

(iii) *If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$, $X_B < 1$ and $X_C \geq 1$, then $v^* < \Phi_B \cdot (1 - X_B/2)$;*
(iv) *If $(X_B, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$, and $X_B, X_C < 1$, then*

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C / (\Phi_B - v^*)}}{2X_B}\right] > \frac{\Phi_B X_B}{2}.$$

(v) *If $(X_B, X_C) \in \mathcal{R}_{2B}(\Phi_B, \Phi_C)$ and $X_C \geq 1$, then*

$$v^* < \Phi_B \cdot \left[1 - \frac{X_B}{2\sqrt{\Phi_B X_B X_C / \Phi_C}}\right]; \text{ and,}$$

(vi) *If $(X_B, X_C) \in \mathcal{R}_{2B}(\Phi_B, \Phi_C)$, $X_C < 1$ and $X_B + X_C \geq 1$, then*

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C / (\Phi_B - v^*)}}{2X_B}\right] > \frac{\Phi_B X_B}{2\sqrt{\Phi_B X_B X_C / \Phi_C}}.$$

We present the proof of Theorem 2 in Appendix B, for ease of presentation. In place of the proof, we devote the remainder of this section to developing the intuition about this positive result.

First, we explain the significance of the value v^* defined in the claim of Theorem 2. Observe that v^* is precisely the battlefield value that satisfies $(\Phi_B - v^*)/X_B = \Phi_C/X_C$. Thus, when $X_B + X_C \geq 1$ and the point (X_B, X_C) is nominally in one of the regions $\mathcal{R}_{1B}(\Phi_B, \Phi_C)$ or $\mathcal{R}_{2B}(\Phi_B, \Phi_C)$, player B can concede a battlefield of value $v_b = v^* + \epsilon$, $\epsilon \to 0^+$, to alter the regions in such a way that (X_B, X_C) is in either $\mathcal{R}_{1C}(\Phi_B - v_b, \Phi_C)$ (when $X_C \geq 1$) or $\mathcal{R}_{2C}(\Phi_B - v_b, \Phi_C)$ (when $X_C < 1$). Note that if $X_B + X_C < 1$ instead, then the concession of the battlefield of value v_b satisfies $(X_B, X_C) \in \mathcal{R}_3(\Phi_B - v_b, \Phi_C)$, and does not offer any benefit to B.

Next, we consider simulation results identifying the parameter regime in which our conditions for beneficial battlefield concessions hold. In Figure 3, we plot player B's optimal battlefield concession, where the players' budgets are normalized such that $X_A = 1$, and $X_B, X_C \in [0, 1.2]$. In each of the panels, the cumulative values of battlefields in the two fronts are as follows: Figure 3a has $\Phi_B = 1, \Phi_C = 0.5$, Figure 3b has $\Phi_B = 1, \Phi_C = 1$, and Figure 3c has $\Phi_B = 1, \Phi_C = 2$. Player B has no beneficial concession in the white area. The various regions $\mathcal{R}_j$, defined in Fact 2, are divided by solid, black lines. The coloured areas' labels coincide with the conditions identified in the claim of Theorem 2.

As seen in Figure 3 and the definitions of Conditions (i)–(vi), the set of beneficial battlefield concessions we identify appear in regions where the ratio between the cumulative value of the battlefields in front $\mathcal{B}_B$ and B's initial budget endowment, Φ_B/X_B, is greater than the ratio Φ_C/X_C, and the players B and C together possess enough budget to force A to prioritize one of her General Lotto games over the other (i.e., $\Phi_B/X_B > \Phi_C/X_C$ and $X_B + X_C \geq 1$). In such scenarios, player

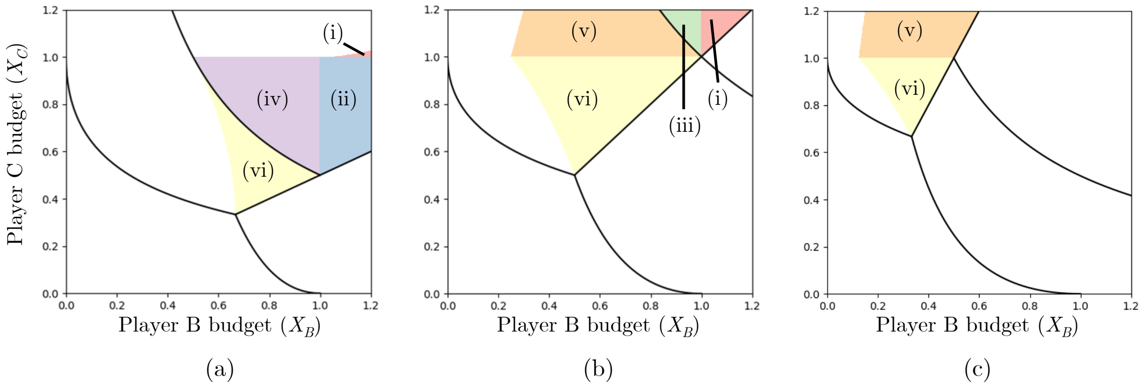


Fig. 3: *Existence of beneficial battlefield concessions for player B.* The coloured regions correspond with our sufficient conditions for beneficial battlefield concessions for player B under normalized player budgets (i.e., $X_A = 1$), for $X_B, X_C \in [0, 1.2]$ and (a) $\Phi_B = 1, \Phi_C = 0.5$, (b) $\Phi_B = 1, \Phi_C = 1$, and (c) $\Phi_B = 1, \Phi_C = 2$. The white area coincide with the points (X_B, X_C) for which our sufficient conditions are not met. The solid, black lines divide the (X_B, X_C)-plane into the various regions, where $\mathcal{R}_{1C}$, $\mathcal{R}_{2C}$ and $\mathcal{R}_3$ are as labelled in plot (a), $\mathcal{R}_{2B}$ is at the top left of each plot (not labelled), and $\mathcal{R}_{1B}$ does not appear. In each of the coloured areas, it is beneficial for player B to concede a battlefield of value v^*, and the different coloured areas' labels coincide with Conditions (i)–(vi), all of which are defined in the claim of Theorem 2.

A primarily pursues her General Lotto game against B in the nominal game. By conceding enough battlefield value $v \in [0, \Phi_B]$ such that $(\Phi_B - v)/X_B < \Phi_C/X_C$, player B can force A to prioritize her game against C instead. Interestingly, if the difference between Φ_B/X_B and Φ_C/X_C is moderate, then the gains from shifting A's attention outweigh the loss of the forfeited battlefield's value. If the difference between Φ_B/X_B and Φ_C/X_C is too high, however, then too much value v must be conceded by B to minimize her conflict with A, and the gains will not outweigh the losses.

In Section 1.3, we briefly describe the variant of the three-player General Lotto game studied in [16]. Recall that, in their variant, the players B and C have the opportunity to negotiate an alliance which entails a unilateral transfer of budgetary resources in Stage 0 of the game, and that cases are identified in which forming an alliance is mutually beneficial for B and C. The results in [16] suggest that mutually beneficial alliances only occur when the difference between the ratios Φ_B/X_B and Φ_C/X_C is sufficiently large. In contrast, our findings show that beneficial battlefield concessions only exist when the ratios Φ_B/X_B and Φ_C/X_C are close. This comparison suggests that, if there are significant asymmetries in the players' strengths relative to the values of their respective contests, then cooperative mechanisms, such as alliances, provide strategic advantages; meanwhile, if differences in players' relative strengths are small, then unilateral mechanisms such as battlefield concessions prevail.

4 Conclusions and Future Work

In this paper, we considered the viability of "concessions" as a component of strategic decision-making in adversarial environments. We considered two types of concessions: *budgetary concessions*, where a competitor voluntarily reduces one's resource budget, and *battlefield concessions*, where a player voluntarily forfeits a certain prize to her adversary. Intuitively, concessions should not offer strategic advantages as they weaken one's competitive position. However, we demonstrated that they do offer benefits if made correctly. We studied concessions under the framework of General Lotto games, where we showed that neither type of concession offers benefits under the two-player model. However, in settings where two players compete against a common adversary, we showed that one of the two players can often improve her payoff by conceding a battlefield to the adversary.

This work provides several avenues for future study. First, we have shown that conceding battlefields is beneficial when General Lotto games are the underlying model of conflict. However, we suspect this phenomenon is robust to larger classes of models, e.g., Tullock and other contest success functions. Second, considering a richer setting wherein both players can simultaneously make concessions to the adversary opens questions of what strategic outcomes are possible. Finally, though we have studied concessions as a strategic component in two- and three-player settings, broader forms of strategic pre-commitments and more general interaction networks could be considered.

A Proof of Theorem 1

Proof. The proof amounts to showing that player B's payoff is nonincreasing for any budgetary concession $x \in (0, X_B]$ such that $(X_B - x, X_C)$ is in any of the regions $\mathcal{R}_j$.

We first consider the scenario where $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$. Recall that, in this scenario, player A commits no budget to the battlefields in the front $\mathcal{B}_B$. Thus, player B's payoff before the concession is Φ_B, the highest possible payoff. Furthermore, $(X_B - x, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$ can only hold if $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B, \Phi_C)$ as well, since the value $1 - \sqrt{\Phi_C(X_B - x)X_C/\Phi_B}$ is increasing in x. If $(X_B - x, X_C) \in \mathcal{R}_4(\Phi_B, \Phi_C)$, then any budgetary concession $x' < x$ would be in either $\mathcal{R}_{1C}$ or $\mathcal{R}_{2C}$, since $X_B + X_C \geq 1$ in $\mathcal{R}_4$, while any budgetary concession $x' > x$ would be in either $\mathcal{R}_{1B}$, $\mathcal{R}_{2B}$ or $\mathcal{R}_3$. Thus, conceding any amount $x' < x$ would guarantee B greater payoff since $X_{A,B} = 1 - X_{A,C}$, and $X_{A,C} = 1$ in $\mathcal{R}_{1C}$ and $X_{A,C} > X_C$ in $\mathcal{R}_{2C}$ whereas $X_{A,C} \in [\max\{0, 1 - X_B\}, \min\{1, X_C\}]$ in $\mathcal{R}_4$. Further, conceding any amount $x' > x$ cannot guarantee B greater payoff since $X_{A,B} = 1$ in $\mathcal{R}_{1B}$, and $X_{A,B} > X_B$ in $\mathcal{R}_{2B}$ and $\mathcal{R}_3$, whereas $X_{A,B} \in [\max\{0, 1 - X_C\}, \min\{1, X_B\}]$ in $\mathcal{R}_4$. In all other regions, we show that player B's payoff is strictly decreasing in x by checking the partial derivative with respect to $x \geq 0$:

If $(X_B - x, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ and $X_B - x > 1$, then

$$\frac{\partial}{\partial x} \Phi_B \left[1 - \frac{1}{2(X_B - x)}\right] = -\frac{\Phi_B}{2(X_B - x)^2} < 0.$$

Else, if $(X_B - x, X_C) \in \mathcal{R}_{1B}(\Phi_B, \Phi_C)$ and $X_B - x \leq 1$, then

$$\frac{\partial}{\partial x} \frac{\Phi_B(X_B - x)}{2} = -\frac{\Phi_B}{2} < 0.$$

If $(X_B - x, X_C) \in \mathcal{R}_{2B}(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x} \frac{\Phi_B(X_B - x)}{2\sqrt{\frac{\Phi_B(X_B - x)X_C}{\Phi_C}}} = -\frac{\Phi_B}{4\sqrt{\frac{\Phi_B(X_B - x)X_C}{\Phi_C}}} < 0.$$

If $(X_B - x, X_C) \in \mathcal{R}_{2C}(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x} \Phi_B \left[1 - \frac{1 - \sqrt{\frac{\Phi_C(X_B - x)X_C}{\Phi_B}}}{2(X_B - x)}\right] = -\frac{\Phi_B \left[2 - \sqrt{\frac{\Phi_C(X_B - x)X_C}{\Phi_B}}\right]}{4(X_B - x)^2} < 0,$$

which is strictly negative as the condition $1-\sqrt{\Phi_C(X_B-x)X_C/\Phi_B} \geq 0$ must hold in $\mathcal{R}_{2C}$. Finally, if $(X_B - x, X_C) \in \mathcal{R}_3(\Phi_B, \Phi_C)$, then

$$\frac{\partial}{\partial x} \frac{\Phi_B(X_B-x)}{2\frac{\sqrt{\Phi_B(X_B-x)}}{\sqrt{\Phi_B(X_B-x)}+\sqrt{\Phi_C X_C}}} = -\frac{\Phi_B}{2} - \frac{\Phi_B\sqrt{\Phi_C X_C}}{4\sqrt{\Phi_B(X_B-x)}} < 0.$$

This concludes the proof. □

B Proof of Theorem 2

Before presenting the proof, we note that, in the case of battlefield concessions, we can disregard the scenario when $(X_B, X_C) \in \mathcal{R}_4(\Phi_B - v, \Phi_C)$, for any $v \in [0, \Phi_B]$. To see why, consider a battlefield concession of value v such that (X_B, X_C) is in $\mathcal{R}_4$, i.e., $X_B + X_C \geq 1$ and $(\Phi_B - v)/X_B = \Phi_C/X_C$. Observe that by conceding a battlefield of value slightly greater than v (i.e., $v' = v + \epsilon$ for $\epsilon \to 0^+$), player B obtains strictly higher payoff as (X_B, X_C) now falls in region $\mathcal{R}_{1C}$ (if $X_C \geq 1$) or region $\mathcal{R}_{2C}$ (if $X_C < 1$). Thus, in the following proof, we assume that any point (X_B, X_C) with $X_B + X_C \geq 1$ will transit directly from $\mathcal{R}_{1B}(\Phi_B - v, \Phi_C)$ (if $X_B \geq 1$) or $\mathcal{R}_{2B}(\Phi_B - v, \Phi_C)$ (if $X_B < 1$), to $\mathcal{R}_{1C}(\Phi_B - v, \Phi_C)$ (if $X_C \geq 1$) or $\mathcal{R}_{2C}(\Phi_B - v, \Phi_C)$ (if $X_C < 1$), as v is increased, without first passing through $\mathcal{R}_4$.

Proof. The proof amounts to verifying that the conditions laid out in the claim guarantee that player B's payoff after the battlefield concession is greater than her payoff in the nominal three-player General Lotto game. Before we continue, it is critical to note that v^* as defined in the claim is precisely the value that satisfies $(\Phi_B - v^*)/X_B = \Phi_C/X_C$. Thus, for $(X_B, X_C) \in \mathcal{R}_{1B} \cup \mathcal{R}_{2B}$ and $X_B + X_C \geq 1$, the battlefield concession of value v^* satisfies $(X_B, X_C) \in \mathcal{R}_{1C} \cup \mathcal{R}_{2C}$. We present the remainder of the proof in parts corresponding with each of the conditions in the claim.
Conditions (i): The point (X_B, X_C) is nominally in the region $\mathcal{R}_{1B}(\Phi_B, \Phi_C)$ with $X_B \geq 1$, and, thus, player B's nominal payoff is $\Phi_B \cdot (1 - 1/2X_B)$. Since $X_C \geq 1$, the battlefield concession of value v^* satisfies $(X_B, X_C) \in \mathcal{R}_{1C}(\Phi_B - v^*, \Phi_C)$, and player B's resulting payoff is $\Phi_B - v^*$. It follows that the battlefield concession of value v^* benefits player B if

$$\Phi_B - v^* > \Phi_B \cdot \left(1 - \frac{1}{2X_B}\right).$$

Rearranging the above inequality gives the condition in the claim.
Condition (ii): Once again, player B's nominal payoff is $\Phi_B \cdot (1 - 1/2X_B)$. Since $X_C < 1$ and $X_B + X_C \geq 1$, the battlefield concession of value v^* satisfies $(X_B, X_C) \in \mathcal{R}_{2C}(\Phi_B - v^*, \Phi_C)$, and player B's resulting payoff is $(\Phi_B - v^*) \cdot [1 - (1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)})/(2X_B)]$. It follows that the battlefield concession of value v^* benefits player B if

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)}}{2X_B}\right] > \Phi_B \cdot \left(1 - \frac{1}{2X_B}\right).$$

Condition (iii): Observe that this condition resembles Condition (i), except that $X_B < 1$. Thus, the only difference is that player B's nominal payoff is $\Phi_B X_B/2$. It follows that the battlefield concession of value v^* benefits player B if

$$\Phi_B - v^* > \frac{\Phi_B X_B}{2}.$$

Rearranging the above inequality gives the condition in the claim.
Condition (iv): Observe that this condition resembles Condition (iii), except that $X_C < 1$. Thus, the only difference is that player B's resulting payoff is $(\Phi_B - v^*) \cdot [1 - (1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)})/(2X_B)]$, as in Condition (ii). It follows that the battlefield concession of value v^* benefits player B if

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)}}{2X_B}\right] > \frac{\Phi_B X_B}{2\sqrt{\Phi_B X_B X_C/\Phi_C}}.$$

Condition (v): The point (X_B, X_C) is nominally in the region $\mathcal{R}_{2B}(\Phi_B, \Phi_C)$, and, thus, player B's nominal payoff is $\Phi_B \cdot X_B/(2\sqrt{\Phi_B X_B X_C/\Phi_C})$ since $X_B < 1$ must hold in $\mathcal{R}_{2B}$. Since $X_C \geq 1$, player B's resulting payoff after the battlefield concession of value v^* is $\Phi_B - v^*$, as in Condition (i). It follows that the battlefield concession of value v^* benefits player B if

$$\Phi_B - v^* > \frac{\Phi_B X_B}{2\sqrt{\Phi_B X_B X_C/\Phi_C}}.$$

Rearranging the above inequality gives the condition in the claim.
Condition (vi): This condition resembles Condition (iv), except that $X_C < 1$. Thus, the only difference is that player B's resulting payoff is $(\Phi_B - v^*) \cdot [1 - (1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)})/(2X_B)]$, as in Condition (ii). It follows that the battlefield concession of value v^* benefits player B if

$$(\Phi_B - v^*) \cdot \left[1 - \frac{1 - \sqrt{\Phi_C X_B X_C/(\Phi_B - v^*)}}{2X_B}\right] > \frac{\Phi_B X_B}{2\sqrt{\Phi_B X_B X_C/\Phi_C}}.$$

This concludes the proof. □

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